# A New Class of Non-topological Solitons

## Joshua A. Frieman[1,3,4] and Bryan W. Lynn[1,2]

[1] *Stanford Linear Accelerator Center*

*Stanford University, Stanford, CA 94309*

[2] *Department of Physics*

*Stanford University, Stanford, CA 94305*

[3] *Institute for Theoretical Physics*

*University of California, Santa Barbara, CA 93106*

[4] *Fermi National Accelerator Laboratory*

*P.O. Box 500, Batavia, IL 60510*

## Abstract

We construct a new class of non-topological solitons in renormalizable scalar field theories with non-linear self-interactions. For large charge $Q$, the soliton mass increases linearly with $Q$, *i.e.*, the soliton mass density is approximately independent of charge. Such objects could be naturally produced in a phase transition in the early universe or in the decay of superconducting cosmic strings.

## I. Introduction

Non-topological solitons (hereafter, NTS's) are solutions of classical field theories which are stable by virtue of a conserved Noether charge carried by fields confined to a finite region of space[1-6]. Recently, these solutions have been studied under the guise of $Q$ balls[7], cosmic neutrino balls[8], quark nuggets[9], and soliton stars[10], and a scenario for producing them in a phase transition in the early universe has been suggested[11]. The simplest renormalizable theory with NTS solutions is an unbroken global $U(1)$ theory of two coupled scalar fields. In this paper, we study the classical NTS solutions of this theory in detail. We will confine our discussion to NTS masses $M \ll m_{pl}^3/m^2$ (where $m$ is the mass of the scalar fields), so that gravity can be neglected. The inclusion of gravitational effects for large mass solutions has been studied elsewhere by one of us[12].

Consider the Lagrangian for a real scalr field $\sigma$ and a complex scalar $\Phi$,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + |\partial_\mu \Phi|^2 - U(|\Phi|, \sigma). \tag{1}$$

The most general interaction invariant under the discrete symmetry $\sigma \to -\sigma$ and the global $U(1)$ symmetry $\Phi \to e^{i\alpha}\Phi$ is

$$U(|\Phi|, \sigma) = \frac{\lambda}{4}(\sigma^2 - \sigma_0^2)^2 + m_{\Phi\Phi}^2 |\Phi|^2 + g^2|\Phi|^4 + \frac{m_{\Phi\sigma}^2}{\sigma_0^2}|\Phi|^2\sigma^2 \tag{2}$$

We are interested in the case $\sigma_0^2 > 0$, so that the discrete symmetry is spontaneously broken in the ground state, $< \sigma >= \pm\sigma_0$. Although this leads to the familiar domain wall problem[13] (if $\sigma_0 > 1 MeV$), we assume this can be circumvented, e.g., by adding explicit discrete symmetry-breaking terms[14]. (Since these terms can be very small, we neglect them in our analysis.) Alternatively, we may consider $\sigma$ as the real component of a complex field whose vacuum expectation value breaks a global $U(1)'$ symmetry; such a theory has no stable domain walls, but may have stable vortices (cosmic strings). In either case, the $U(1)$ symmetry carried by $\Phi$ is unbroken, so there is a conserved current

$$J_\mu = -i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) \tag{3}$$

and an associated conserved charge

$$Q = \int d^3r J_0. \tag{4}$$

(The conserved charge stabilizes the NTS against decay.) In vacuum, the particle masses are given by

$$m_\sigma^2 = 2\lambda\sigma_0^2, \qquad m_\Phi^2 = m_{\Phi\Phi}^2 + m_{\Phi\sigma}^2. \tag{5}$$

Spherically symmetric NTS solutions for the theory of Eqns. (1-2) were first studied by Friedberg, Lee and Sirlin[3] for the special case $m_{\Phi\Phi} = g = 0$. For large $Q$, they are characterized by an interior "false vacuum" region where $\sigma \simeq 0$, surrounded by a thin domain wall where $\sigma$ rapidly approaches its ground state value $\sigma = \sigma_0$. In the NTS interior, the potential energy density in $\sigma$ is balanced by the pressure of the massless $\Phi$ charges, which are confined by the mass gap $\sim m_{\Phi\sigma}$ at the domain wall.

Here, we consider a more general class of solutions. In particular, we show below that inclusion of the repulsive $|\Phi|^4$ self-interaction completely changes the behavior of the NTS's, altering the mass-radius and mass-charge relations. In the next Section we discuss approximate analytic NTS solutions, valid for large $Q$. In Section III, we construct numerical solutions, which are useful for displaying the behavior at small charge. We consider the spectrum of NTS excitations (phonons and particles) in section IV. In the Conclusion, we speculate on the place of NTS's in particle physics models and briefly outline how such objects might be produced in the early universe.

## II. Analytic NTS Solutions

From Eqns. (3-4), a classical solution $\Phi(\vec{r}, t)$ must be time-dependent for the system to have non-zero charge. The lowest energy solutions are of the spherically symmetric form[1-4]

$$\Phi(\vec{r}, t) = \frac{e^{i\omega t}\varphi(r)}{\sqrt{2}}, \qquad \sigma(\vec{r}, t) = \sigma(r). \tag{6}$$

It proves convenient to define the rescaled field variables,

$$\tilde{\varphi} = \frac{\varphi}{\sigma_0}, \qquad \tilde{\sigma} = \frac{\sigma}{\sigma_0}, \tag{7}$$

spatial coordinate,

$$\vec{y} = g\sigma_0\vec{r}, \tag{8}$$

masses

$$\frac{\tilde{m}_\sigma}{m_\sigma} = \frac{\tilde{m}_{\Phi\Phi}}{m_{\Phi\Phi}} = \frac{\tilde{m}_{\Phi\sigma}}{m_{\Phi\sigma}} = \frac{\tilde{w}}{w} = \frac{1}{g\sigma_0}, \tag{9}$$

and coupling constant,

$$\tilde{\lambda} = \frac{\lambda}{g^2} = \frac{\tilde{m}_\sigma^2}{2}. \tag{10}$$

The energy functional for the solution (6) can then be written

$$M = \frac{\tilde{M}\sigma_0}{g} = \frac{4\pi\sigma_0}{g} \int_0^\infty y^2 dy \left\{ \frac{1}{2}(\partial_y\tilde{\varphi})^2 + \frac{1}{2}(\partial_y\tilde{\sigma})^2 + \frac{1}{2}\tilde{w}^2\tilde{\varphi}^2 + \tilde{U}(\tilde{\varphi},\tilde{\sigma}) \right\}, \tag{11}$$

where

$$\tilde{U}(\tilde{\varphi},\tilde{\sigma}) = \frac{1}{2}(\tilde{m}_{\Phi\Phi}^2 + \tilde{m}_{\Phi\sigma}^2\tilde{\sigma}^2)\tilde{\varphi}^2 + \frac{\tilde{\varphi}^4}{4} + \frac{\tilde{\lambda}}{4}(\tilde{\sigma}^2 - 1)^2 \tag{11a}$$

is the rescaled potential. The soliton charge is

$$Q = \frac{4\pi}{g^2} \int_0^\infty dy y^2 \tilde{w}\tilde{\varphi}^2 \equiv \frac{\tilde{Q}}{g^2}. \tag{12}$$

The rescaled field equations are

$$\frac{d^2\tilde{\varphi}}{dy^2} + \frac{2}{y}\frac{d\tilde{\varphi}}{dy} + (\tilde{w}^2 - \tilde{m}_{\Phi\Phi}^2)\tilde{\varphi} = \tilde{\varphi}(\tilde{m}_{\Phi\sigma}^2\tilde{\sigma}^2 + \tilde{\varphi}^2) \tag{13}$$

$$\frac{d^2\tilde{\sigma}}{dy^2} + \frac{2}{y}\frac{d\tilde{\sigma}}{dy} - \tilde{\lambda}\tilde{\sigma}(\tilde{\sigma}^2 - 1) = \tilde{m}_{\Phi\sigma}^2\tilde{\varphi}^2\tilde{\sigma}. \tag{14}$$

The method of constructing NTS solutions is, in principle, straightforward: they are uniquely determined by specifying the parameters $\tilde{w}$, $\tilde{m}_{\Phi\Phi}$, $\tilde{m}_{\Phi\sigma}$ and $\tilde{\lambda}$, and by imposing the boundary conditions $\tilde{\varphi}(y) \to 0$, $\tilde{\sigma}(y) \to 1$ as $y \to \infty$, and $d\tilde{\varphi}/dy, d\tilde{\sigma}/dy \to 0$ as $y \to 0$. One then solves Eqns. (13-14) subject to the constraint that $\tilde{M}$, the rescaled NTS mass, is

4

minimized; this also fixes the soliton charge $\tilde{Q}$. We note that each such solution actually yields a two-parameter family of NTS's, since $g$ and $\sigma_0$ have been scaled out.

To gain insight into the nature of the NTS solutions, one can think of Eqns. (13) and (14) as the equations of motion of a particle rolling in a two-dimentional potential $V_{eff}(\tilde{\varphi}, \tilde{\sigma})$, with "space coordinates" $(\tilde{\varphi}, \tilde{\sigma})$ and "time coordinate" $y$. The particle is subject to a friction force which varies as $y^{-1}$. The effective potential for this mechanical analogy is

$$V_{eff}(\tilde{\varphi}, \tilde{\sigma}) = \frac{1}{2}\tilde{w}^2\tilde{\varphi}^2 - \tilde{U}(\tilde{\varphi}, \tilde{\sigma}) \tag{15}$$

and is shown in Fig. 1 for a particular choice of parameters. At $y = 0$, the NTS center, the particle starts at rest, while at late "times", $y \to \infty$ (*i.e.*, far outside the NTS), it must approach the vacuum state $\tilde{\sigma} = 1, \tilde{\varphi} = 0$ on the ridge of the potential. Since the particle rolls with friction, it must start out at a point higher on the potential saddle, where $\tilde{\sigma} < 1$, $\tilde{\varphi} > 0$. Clearly, then, a necessary condition for the existence of a solution is

$$\mathrm{max}V_{eff}(\tilde{\varphi}, \tilde{\sigma}) \geq V_{eff}(0,1) = 0. \tag{16}$$

Suppose this condition is true for some choice of the parameters, and then imagine decreasing $\tilde{w}$ while keeping the masses and couplings fixed. From Eqn. (15), the inequality (16) then approaches equality so that, for a solution to exist, the friction term must become negligible. But approximate solutions in which the friction term can be consistently dropped are easy to construct. They are of the form

$$\tilde{\varphi}(y) \simeq \begin{cases} \tilde{\varphi}_c & , \quad y < Y \\ 0 & , \quad y > Y \end{cases} \tag{17a}$$

$$\tilde{\sigma}(y) \simeq \begin{cases} \tilde{\sigma}_c & , \quad y < Y \\ 1 & , \quad y > Y. \end{cases} \tag{17b}$$

Here, the particle stays close to its initial position $(\tilde{\varphi}_c, \tilde{\sigma}_c)$ until a large time, $y = Y$, when it suddenly rolls down the potential and asymptotically comes to rest at the vacuum state. For the ansatz (17), the friction terms $y^{-1}d\tilde{\varphi}/dy$, $y^{-1}d\tilde{\sigma}/dy$ would only be appreciable

where the field gradients are large, at $y \sim Y$; however, if $Y$ is sufficiently large, these terms are suppressed.

The solution above describes large, thin-walled solitons of physical radius $R = Y/g\sigma_0$, with negligible surface energy. We call these large $Q$ solitons "bag" solutions by analogy with hadron models. One feature of these bag solutions should be emphasized: by construction [Eqns. (8-10)], they only exist for $g \neq 0$, i.e., for non-zero $|\Phi|^4$ coupling. To see this, suppose we instead define a different coordinate rescaling $\vec{x} = \sigma_0 \vec{r}$, and correspondingly define rescaled parameters $\bar{m}_{\Phi\Phi}$, $\bar{m}_{\Phi\sigma}$, $\bar{w}$, and $\bar{\lambda}$, obtained by setting $g = 1$ in Eqns. (9) and (10). Then Eqn. (13) becomes

$$\frac{d^2\tilde{\varphi}}{dx^2} + \frac{2}{x}\frac{d\tilde{\varphi}}{dx} + (\bar{w}^2 - \bar{m}_{\Phi\Phi}^2)\tilde{\varphi} = \tilde{\varphi}(\bar{m}_{\Phi\sigma}^2 \tilde{\sigma}^2 + g^2\tilde{\varphi}^2). \tag{18}$$

In the limit $g \to 0$, the solution (17a) becomes inconsistent. Rather, for $g = 0$ and $\tilde{\sigma} = \tilde{\sigma}_c = $ const. Eqn. (18) is just the linear equation for spherical waves, with solution[3]

$$\tilde{\varphi} = \frac{\tilde{\varphi}_c}{x}\sin\left[(\bar{w}^2 - \bar{m}_{\Phi\Phi}^2 - \bar{m}_{\Phi\sigma}^2\tilde{\sigma}_c^2)^{1/2}x\right]. \tag{19}$$

For large $Q$, it thus turns out that, unless $g$ is very small, $g \ll Q^{-1/4}$, the solutions with non-zero $g$ are qualitatively different from those with $g = 0$. In particular, at large $Q$, for $g = 0$, the central amplitude $\tilde{\varphi}_c$ grows as a power of $Q$ while, for $g \neq 0$, $\tilde{\varphi}_c$ approaches a constant, independent of $Q$ (see below). This arises because the $g^2|\Phi|^4$ contribution to the NTS energy density suppresses the amplitude of $\Phi$. We will compare the cases $g = 0$, $g \neq 0$ more fully in our numerical work below.

Inserting the ansatz (17) in Eqns. (11) and (12), we find the rescaled charge

$$\tilde{Q} = \frac{4\pi}{3}Y^3\tilde{w}\tilde{\varphi}_c^2, \tag{20}$$

and NTS mass

$$\tilde{M} = \frac{4\pi}{3}Y^3\{\frac{1}{2}\tilde{w}^2\tilde{\varphi}_c^2 + \tilde{U}(\tilde{\varphi}_c, \tilde{\sigma}_c)\}, \tag{21}$$

6

Using Eqn. (20) to eliminate $\tilde{w}$, Eqn. (21) can be rewritten

$$\tilde{M} = \frac{Q^2}{2\tilde{V}\tilde{\varphi}_c^2} + \tilde{V}\tilde{U}(\tilde{\varphi}_c, \tilde{\sigma}_c),$$
(22)

where $\tilde{V} = \frac{4\pi}{3}Y^3$ is the rescaled NTS volume. Minimizing the energy with respect to $\tilde{V}$ at fixed charge gives

$$\tilde{V} = \frac{\tilde{Q}}{\tilde{\varphi}\sqrt{2\tilde{U}(\tilde{\varphi}_c, \tilde{\sigma}_c)}}, \qquad \tilde{M} = \frac{\tilde{Q}\sqrt{2\tilde{U}(\tilde{\varphi}_c, \tilde{\sigma}_c)}}{\tilde{\varphi}_c}.$$
(23)

If we minimize the mass $\tilde{M}$ (at fixed $\tilde{Q}$) with respect to $\tilde{\varphi}_c$ and $\tilde{\sigma}_c$, we obtain

$$\tilde{\sigma}_c = 0, \qquad \tilde{\varphi}_c = \tilde{\lambda}^{1/4},$$
(24)

so that $\varphi_c^4 = \lambda\sigma_0^4/g^2$. Thus, the NTS energy is

$$\tilde{M} = \tilde{Q}(\tilde{\lambda}^{1/2} + \tilde{m}_{\Phi\Phi}^2)^{1/2} = \tilde{w}_0\tilde{Q}$$
(25)

or

$$M = w_0 Q = Q(\lambda^{1/2}g\sigma_0^2 + m_{\Phi\Phi}^2)^{1/2}.$$
(26)

For Eqn. (23), the NTS charge is

$$Q = w_0\varphi_c^2 V,$$
(27)

so the mass-radius relation is

$$M = w_0^2\varphi_c^2 V = (\lambda\sigma_0^4 + \frac{\lambda^{1/2}\sigma_0^2 m_{\Phi\Phi}^2}{g})V,$$
(28)

i.e., large charge NTS's have approximately constant density, $\rho = w_0^2\varphi_c^2$. We note that these relations, i.e., $M \sim Q, R \sim Q^{1/3}$, are very different from the "free" case $g = m_{\Phi\Phi} = 0$ treated in Ref. 3, where $M \sim Q^{3/4}, R \sim Q^{1/4}$. In particular, the solutions displayed above have a minimum frequency, $w \to w_0$ as $Q \to \infty$, while "free" solitons have $w = \pi/R \sim Q^{-1/4}$. For the self-interacting theory, the NTS solutions are very akin to Q-balls[7].

Non-topological solitons are quantum-mechanically stable if they are the lowest energy configuration of fixed charge. Using Eqn. (5), the stability conditions are

$$M(Q) < Q(m_{\Phi\Phi}^2 + m_{\Phi\sigma}^2)^{1/2} \tag{29}$$

and

$$\frac{d^2 M}{dQ^2} < 0. \tag{30}$$

Eqn. (29) ensures stability against decay into free $\Phi$ particles, while Eqn. (30) expresses stability against fission into smaller soliton fragments.

For large $Q$, the stability condition (29) becomes [from (26)]

$$g\lambda^{1/2}\sigma_0^2 < m_{\Phi\sigma}^2, \tag{31}$$

*independent* of the "bare" mass $m_{\Phi\Phi}$. To check stability against fission, we would need to include the surface terms neglected above. This is easily done: treated perturbatively, the surface energy $4\pi R^2 \mu$ (where $\mu$ is the surface energy per unit area) contributes a term $\sim Q^{2/3}$ to the NTS mass, and therefore $d^2M/dQ^2 < 0$. Thus, the binding energy per unit charge increases monotonically with $Q$, and large $Q$ NTS's do not fission spontaneously. The addition of the surface energy also implies that the stability bound (29) is violated at sufficiently small charge, *i.e.*, there exists a minimum charge $Q_{\min}$ above which NTS's are absolutely stable. Generally, for $Q \sim Q_{\min}$ the surface terms cannot be treated as perturbation, so we defer discussion of this point to the numerical solutions below.

## III. Numerical NTS Solutions

To construct solutions at arbitrary $Q$, we have numerically solved Eqns. (13) and (14) subject to the boundary conditions noted in the previous Section. The program uses the parameters $\tilde{w}$, $\tilde{m}_{\Phi\Phi}$, $\tilde{m}_{\Phi\sigma}$ and $\tilde{\lambda}$ as inputs and returns $\tilde{M}$ and $\tilde{Q}$ as outputs. It is useful to write the energy functional as

$$\tilde{M} = \tilde{E}_K + \tilde{E}_P + \tilde{E}_N, \tag{32}$$

8

where the gradient energy

$$\tilde{E}_K = 4\pi \int y^2 dy \left[ \frac{1}{2}(\partial_y \tilde{\varphi})^2 + \frac{1}{2}(\partial_y \tilde{\sigma})^2 \right],$$ (33)

the "potential" energy

$$\tilde{E}_P = 4\pi \int y^2 dy \left[ \tilde{U}(\tilde{\varphi}, \tilde{\sigma}) - \frac{\tilde{w}^2 \tilde{\varphi}^2}{2} \right]$$ (34)

and the "number" energy

$$\tilde{E}_N = 4\pi \int y^2 dy \tilde{w}^2 \tilde{\varphi}^2 = \tilde{w}\tilde{Q}.$$ (35)

For the analytic solutions of the previous Section, $\tilde{M} \simeq \tilde{E}_N$. More generally, one can show that the NTS solutions must satisfy a virial theorem,[3]

$$\tilde{E}_P = -\frac{1}{3}\tilde{E}_K.$$ (36)

This relation is used to check the accuracy of the numerical solutions: the program iterates from a trial solution until the quantity

$$\frac{\tilde{E}_P + \frac{1}{3}\tilde{E}_K}{\tilde{E}_P}$$ (37)

is sufficiently small, typically less than $10^{-4}$ or so.

In Fig. 2, we show two NTS solutions with parameters $\tilde{m}_{\Phi\Phi} = 0$, $\tilde{m}_{\Phi\sigma} = 1$, $\tilde{\lambda} = 1/9$ (assuming $g \neq 0$). The solid curves show $\tilde{\varphi}(y)$, $\tilde{\sigma}(y)$ for $\tilde{w}^2 = 0.38$, with charge $\tilde{Q} = 3.1 \times 10^4$ and mass $\tilde{M} = 2.0 \times 10^4$; in this case, $\tilde{w}_0^2 = \tilde{\lambda}^{1/2} = 0.33$, so this is a bag solution with $\tilde{w} \simeq \tilde{w}_0$ and $\tilde{M} \simeq \tilde{w}\tilde{Q}$. The dashed curves show the solution for $\tilde{w}^2 = 0.8$, which has a charge $\tilde{Q} = 74.5$ and mass $\tilde{M} = 72.2$. For this case, the potential and surface energies are important and, as one sees from the figure, the solution is not at all bag-like. In particular, here $\tilde{\sigma}(0) = 0.5$ instead of zero. As $\tilde{w}$ is further increased, eventually we reach a charge $\tilde{Q}_{\mathrm{min}}$ where the stability condition (29) breaks down. For the parameters above, this occurs for $\tilde{w}^2 = 0.92$, with $\tilde{Q}_{\mathrm{min}} = 45.1$ and $\tilde{M}_{\mathrm{min}} = 45.0$.

It is interesting to consider what happens for $\tilde{Q} < \tilde{Q}_{\min}$. Although such NTS's have positive binding energy, and are thus quantum mechanically unstable, classical solutions still exist. As $\tilde{w}^2$ is increased from 0.92 (its value at $\tilde{Q}_{\min}$), the charge and mass first continue to drop, but then $\tilde{M}(\tilde{Q})$ reaches a cusp at $\tilde{Q}_{\text{crit}} = 39.4$, $\tilde{M}_{\text{crit}} = 39.5$, $\tilde{w}^2_{\text{crit}} = 0.97$. For $\tilde{w} > \tilde{w}_{\text{crit}}$, the mass and charge *grow* with increasing $\tilde{w}$ and, as $\tilde{w}$ approaches $\tilde{m}_{\Phi\sigma} = 1$ from below, the NTS energy approaches the free particle energy $\tilde{M}_{\text{free}} = \tilde{m}_{\Phi\sigma}\tilde{Q} = \tilde{Q}$. Thus, a plot of $\tilde{M}(\tilde{Q})$ would show two branches, the upper one with $\tilde{w} > \tilde{w}_{\text{crit}}$, the lower with $\tilde{w} < \tilde{w}_{\text{crit}}$, which join at $\tilde{Q}_{\min}$. (Similar behavior at small $Q$ was found in Ref 3.) Presumably, the upper branch solutions decay either to the lower branch or to free $\Phi$ particles.

For comparison, in Fig. 3 we show NTS solutions with the same parameters as above, but with $g = 0$, *i.e.*, we have $m_{\Phi\Phi} = 0$, $\tilde{m}_{\Phi\sigma} = m_{\Phi\sigma}/\sigma_0 = 1$, $\tilde{\lambda} = \lambda = 1/9$, and we plot $\tilde{\varphi}(x)$, $\tilde{\sigma}(x)$ where $x = \sigma_0 r$ [see discussion around Eqn. (18)]. The solid curves are for the case $\tilde{w}^2 = w^2/\sigma_0^2 = 0.038$, $\tilde{Q} = Q = 9.6 \times 10^3$, $\tilde{M} = M/\sigma_0 = 2.5 \times 10^3$, while the dashed curves are for $w^2 = 0.6$, $Q = 64.1$, $\tilde{M} = 59.4$. It is clear that these solutions are far from bags, but are instead well approximated by the solutions (17b), (19) for large $Q$. For these parameters, the minimum charge for stability is $Q_{\min} = 34.5$, $\tilde{M}_{\min} = 34.5$, which occurs at $\tilde{w}^2 = 0.875$. To see the effect of the $g^2|\Phi|^4$ term graphically, we plot the soliton mass $\tilde{M}(\tilde{M})$ as a function of charge $\tilde{Q}(\tilde{Q})$ for the case $g \neq 0(g = 0)$, in Fig. 4. To compare the curves, we can take $g = 1$ for the model with non-zero $g$, so that we are plotting $M/\sigma_0$ vs. $Q$ in both cases. As expected, at large charge $Q$, the $|\Phi|^4$ term increases the NTS mass. (We have displayed only the lower branch solutions.)

To summarize, for the self-interacting theory, stable NTS solutions exist over the frequency range

$$(\lambda^{1/2}g\sigma_0^2 + m_{\Phi\Phi}^2)^{1/2} < w < (m_{\Phi\sigma}^2 + m_{\Phi\Phi}^2)^{1/2} \tag{38}$$

and for charges greater than a minimum charge $Q_{\min}$ which depends on coupling constants.

## IV. Soliton Excitations

The phenomenology of non-topological solitons is determined in part by the small fluctuations about the NTS ground state. For example, at finite temperature, these fluctuations make an important contribution to the free energy of the NTS and are therefore crucial in determining the phase diagram of the theory. These excitations are of three types: i) $\sigma$ particle and $\Phi$ particle-antiparticle excitations, with energies proportional to the $(\sigma, \Phi)$ masses in the NTS interior; ii) sound waves (phonons), with energies proportional to $R^{-1}$; iii) surface waves. In this Section, we compute the spectrum of NTS fluctuations of types (i) and (ii) for large $Q$. For simplicity, we shall focus on the case $m_{\Phi\Phi} = 0$.

To study small amplitude fluctuations, we define the perturbed variables,

$$\hat{\sigma} = \sigma(r) + \delta\sigma(\vec{r}, t), \tag{39}$$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \hat{\varphi}, \tag{40}$$

where

$$\hat{\varphi} = \begin{pmatrix} \varphi(r) + \delta\varphi_1(\vec{r}, t) \\ \delta\varphi_2(\vec{r}, t) \end{pmatrix}. \tag{41}$$

Here, $\sigma(r)$ and $\varphi(r)$ are the unperturbed NTS solutions of Eqn. (6). Linearizing in $\delta\sigma, \delta\varphi_1, \delta\varphi_2$ and using the unperturbed equations of motion for the background solution yields

$$\left( \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + a_2 \right) \delta\sigma + a_3 \delta\varphi_1 = 0 \tag{42}$$

$$a_3 \delta\sigma + \left( \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + a_1 + a_4 \right) \delta\varphi_1 - 2\omega \frac{\partial}{\partial t} \delta\varphi_2 = 0 \tag{43}$$

$$2\omega \frac{\partial}{\partial t} \delta\varphi_1 + \left( \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + a_1 \right) \delta\varphi_2 = 0. \tag{44}$$

Here, $\omega$ is the frequency of the background soliton, and $a_1, ..., a_4$ are functions of the unperturbed solution,

$$a_1 = \frac{m_{\Phi\sigma}^2}{\sigma_0^2} \sigma^2 + g^2\varphi^2 - \omega^2 \tag{45}$$

11

$$a_2 = \frac{m_{\Phi\sigma}^2}{\sigma_0^2}\varphi^2 + \lambda(\sigma^2 - \sigma_0^2) + 2\lambda\sigma^2 \tag{46}$$

$$a_3 = \frac{2m_{\Phi\sigma}^2}{\sigma_0^2}\varphi\sigma \tag{47}$$

$$a_4 = 2g^2\varphi^2. \tag{48}$$

For arbitrary charge, one can in principle solve Eqns. (42-44) for the normal modes of the NTS; the solutions involve spherical Bessel functions and Legendre polynomials, with suitable boundary conditions imposed at the NTS surface. We shall instead consider modes of the large $Q$ bags of Section II.

Using the bag solution (24), we have $a_1 = a_3 = 0$, and

$$a_2 = \frac{m_{\Phi\sigma}^2\lambda^{1/2}}{g} - \lambda\sigma_0^2 \tag{49}$$

$$a_4 = 2\omega_0^2 = 2g\lambda^{1/2}\sigma_0^2. \tag{50}$$

Thus, in the large Q limit, the $\delta\sigma$ and $\delta\varphi$ modes decouple. Furthermore, since we are considering the limit of infinite NTS radius, we can approximate the normal modes with plane waves of the form $e^{i(k_0 t - \vec{k}\cdot\vec{x})}$. For the $\delta\varphi$ modes, we then find a dispersion relation of the form

$$k_0^2 = k^2 + 3\omega_0^2 \pm \omega_0(4k^2 + 9\omega_0^2)^{1/2}, \tag{51}$$

where $k^2 = |\vec{k}|^2$. At low momentum, i.e., $k \ll \omega_0$, the lower (acoustic) branch gives $k_0^2 = k^2/3$, which corresponds to the sound speed of a relativistic medium, $c_s^2 = 1/3$. The upper (optical) branch corresponds to $\Phi$ particle-antiparticle excitations, with rest energy $k_0 \simeq \sqrt{6}\omega_0$.

For the $\delta\sigma$ modes, in the bag limit the mode equation (42) is just the free Klein-Gordon equation with mass $m = \sqrt{a_2}$ [see Eqn. (49)]. We note that the stability condition (31) implies $a_2 > 0$; in other words, as one would expect, NTS stability corresponds to non-tachyonic $\delta\sigma$ fluctuations.

12

## V. Conclusion

Although we have studied the simplest theory containing non-topological solitons, many features of the solutions discussed here are universal. For example, for large charge, we found that scalar NTS's have approximately constant mass density, i.e., mass proportional to their charge. This scaling holds whenever the charged field, be it fermion or boson, has a reduced mass in a region of *non-zero* vacuum energy. Thus, the linear scaling of mass with charge is valid for Q-balls, strange nuggets, bag models of hadrons, etc.

Non-topological solitons can arise in any particle theory in which the lightest field carrying a global additively conserved quantum number gets a mass via a Higgs-type mechanism. Many extensions of the standard model, such as technicolor, grand unified theories, left-right symmetric theories and majoron models may have such a structure. Candidates for the requisite charged field include the lightest technibaryon and a massive neutrino. Whether the NTS solutions in such theories are actually stable depends only on ratios of coupling constants. Generally, one finds stability above some minimum charge $Q_{min}$ (if at all); for theories with small coupling constants (of order unity), $Q_{min}$ typically lies in the range between one and $10^4$ or so. For $Q_{min} \leq 1$, we have the intriguing possibility that NTS's are the lowest lying solutions in the particle spectrum.

In passing, let us note that NTS's may also arise in theories with a *multiplicatively* conserved charge. An interesting example is supersymmetry, where $Q$ is identified with R-parity. If the lightest supersymmetric particle is a scalar neutrino (or a Majorana fermion such as the photino), the associated NTS's would be qualitatively similar to those described here. However, since R-parity is multiplicative, sneutrino solitons with $R \geq 2$ can decay by self-annihilation. Supersymmetric NTS's would only survive if the NTS frequency $\omega$ is below the mass threshold of the lightest LSP annihilation channel or if $R_{min} \leq 1$.

How might such objects come into existence? During a phase transition in the early universe, NTS's may condense due to charge fluctuations or due to a charge asymmetry

(i.e., if the universe carries a net $Q$).[11] To study this process in detail, we are at present constructing NTS solutions at finite temperature.[15] Once formed, however, NTS's are generally vulnerable to evaporation[16] above a critical temperature at which they become the lowest free energy state. Whether they survive this evaporation phase is model-dependent.

Another mechanism for NTS formation is the decay of cosmic strings. Suppose $\sigma$ is the real component of a complex field in a $U(1) \times U(1)'$ theory. For certain choices of parameters, this theory is known to have vortex solutions which carry both current and charge.[17] In some cases, the resulting 'vortons' may evolve to a state in which the charged condensate is separated from the current; when this happens, it is likely that a stable NTS is formed. Since cosmic strings can be efficiently produced in phase transitions,[18] NTS's may be produced in abundance.

# REFERENCES

1. G. Rosen, *J. Math. Phys.* **9**, 996 (1968).

2. T.D. Lee and G.C. Wick, *Phys. Rev. D* **9**, 2291 (1974).

3. R. Friedberg, T.D. Lee and A. Sirlin, *Phys. Rev. D* **13**, 2379 (1976), and Nucl. Phys. **B115**, 1, 32 (1976).

4. R. Friedberg and T.D. Lee, *Phys. Rev. D* **15**, 1694 (1977).

5. J. Werle, *Phys. Lett.* **71B**, 367 (1977).

6. T.F. Morris, *Phys. Lett.* **76B**, 337 (1978), **78B**, 87, (1978).

7. S. Coleman, Nucl. Phys. **B262**, 263, 1985. A.M. Safian, S. Coleman and M. Axenides, Nucl. Phys. **B297**, 498, 1988. A.M. Safian, Nucl. Phys. **B304**, 392, 1988.

8. B. Holdom, *Phys. Rev. D* **36**, 1000 (1987).

9. E. Witten, *Phys. Rev. D* **30**, 272 (1984). E. Farhi and A. Jaffe, *Phys. Rev. D* **30**, 2379 (1984). C. Alcock and E. Farhi, *Phys. Rev. D* **32**, 1273 (1985).

10. R. Ruffini and S. Bonazzola, Phys. Rev. **187**, 1767 (1969). J.D. Breit, S. Gupta and A. Zaks, *Phys. Lett.* **140B**, 329 (1984). M. Colpi, S.L. Shapiro and I. Wasserman, Phys. Rev. Lett. **57**, 2485 (1986). C. Alcock, E. Farhi and A. Olinto, Astrophys. J. **310**, 261, (1986). T.D. Lee, *Phys. Rev. D* **35**, 3637 (1987). R. Friedberg, T.D. Lee and Y. Pang, Phys. Rev. **D35**, 3640, 3658 (1987). T.D. Lee and Y. Pang, *Phys. Rev. D* **35**, 3678 (1987). M. Gleiser, Fermilab preprint (1988). M. Gleiser and R. Watkins, ITP preprint (1988). P. Jetzer, Fermilab preprint (1988).

11. J. Frieman, G. Gelmini, M. Gleiser and E. Kolb, Phys. Rev. Lett. **60**, 2101 (1988).

12. B.W. Lynn, SLAC-PUB-4715 (1988).

13. Ya. B. Zel'dovich, I. Yu. Kobzarev and L.B. Okun, Zh. Eksp. Teor. Fiz. **67**, 3 (1974) [Sov. Phys. JETP **40**, 1 (1975)].

14. T.W.B. Kibble, J. Phys. **A9**, 1387 (1976). P. Sikivie, Phys. Rev. Lett. **48**, 1156 (1982). A. Vilenkin, *Phys. Rev. D* **21**, 263 (1985). See also Ref. 11.

15. S. Bahcall, J. Frieman and B. Lynn, Formation of Non-topological Solitons, in preparation.

16. C. Alcock and E. Farhi, ref. 9. J. Frieman, M. Gleiser and A. Olinto, in preparation.

17. E. Witten, Nucl. Phys. **B249**, 557 (1985). R. Davis and P. Shellard, ITP preprint (1988).

18. T.W.B. Kibble, Phys. Rep. **67**, 183 (1980).

**Figure Captions**

1. Effective potential for the mechanical analogy, $V_{eff}(\tilde{\varphi}, \tilde{\sigma})$ [Eqn.(15)].

2. NTS solutions $\tilde{\varphi}(y), \tilde{\sigma}(y)$, for $\tilde{m}_{\Phi\Phi} = 0$, $\tilde{m}_{\Phi\sigma} = 1$, $\tilde{\lambda} = 1/9$, $g \neq 0$. Solid curves: $\tilde{\omega}^2 = 0.38$, $\tilde{Q} = 3.1 \times 10^4$. Dashed curves: $\tilde{\omega}^2 = 0.8$, $\tilde{Q} = 74.5$.

3. NTS solutions $\tilde{\varphi}(x), \tilde{\sigma}(x)$, for same parameters as Fig. 2, but with $g = 0$. Solid curves: $\bar{\omega}^2 = 0.038$, $\bar{Q} = 9.6 \times 10^3$. Dashed curves: $\bar{\omega}^2 = 0.6$, $\bar{Q} = 64.1$.

4. NTS mass $M/\sigma_0$ as a function of charge $Q$. Dashed curve: $g = 0$. Solid curve: $g = 1$. Dotted curve: free particles, $M = Qm_{\Phi\sigma}$.